Folded Localized Excitations and Chaotic Patterns in a (2+1)-Dimensional Soliton System

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Starting from an improved mapping approach and a linear variable separation approach, new families of variable separation solutions (including solitary wave solutions, periodic wave solutions and rational function solutions) with arbitrary functions for the (2+1)-dimensional breaking soliton system are derived. Based on the derived solitary wave solution, we obtain some special folded localized excitations and chaotic patterns.

Key words: Improved Mapping Approach; Variable Separation Approach; Breaking Soliton System; Folded Localized Excitations; Chaotic Patterns.

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1. Introduction

Modern soliton theory is widely applied in many natural sciences [1 – 5] such as chemistry, biology, mathematics, communication, and particularly in almost all branches of physics like fluid dynamics, plasma physics, field theory, optics, and condensed matter physics [6 – 9]. Previously, much efforts have been focused on the single-valued localized excitations, such as solitoffs, dromions, rings, lumps, breathers, instantons, peakons, compactons, localized chaotic, and fractal patterns [10 – 15]. However, there are various complicated phenomena in the real natural world like the usual bubbles on a fluid surface or ocean waves, which are folded and obviously cannot be described by single-valued functions. In [16], Tang and Lou introduced some multi-valued localized excitations to describe folded solitary waves and to define a new type of soliton-foldon. Actually, the simple foldons in lower dimensions can be equivalently called the loop solitons which can be found in many (1+1)-dimensional integrable models [17, 18] and have been applied in some physical branches like quantum theory, string theory, and particle physics [19, 20]. But for these lower-dimensional foldons, we know little on foldons in higher dimensions. In this paper, by using some multi-valued functions, we found some new folded localized excitations in the (2+1)-dimensional breaking soliton system

$$u_{xt} - 4u_{xy}u_x - 2u_{xx}u_y - u_{xxxy} = 0. \qquad (1)$$

Equation (1) was used to describe the (2+1)-dimensional interaction of Riemann waves propagating along the y-axis with long waves propagating along the x-axis [21].

2. New Exact Solutions to the (2+1)-Dimensional Breaking Soliton System

As is well known, to search for the solitary wave solutions of a nonlinear physical model, we can apply different approaches. One of the most efficient methods to find soliton excitations of a physical model is the so-called improved mapping approach. The basic idea of the algorithm is as follows. For a given nonlinear partial differential equation (NPDE), with the independent variables $x = (x_0 = t, x_1, x_2, \cdots, x_m)$ and the dependent variable u, in the form

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \cdots) = 0, \qquad (2)$$

where P is in general a polynomial function of its arguments, and the subscripts denote the partial derivatives, the solution can be assumed to be in the form

$$u = A(x) + \sum_{i=1}^{n} \Big\{ B_i(x)\phi^i[q(x)] + \frac{C_i}{\phi^i[q(x)]} + D_i(x)\phi^{i-1}[q(x)]\sqrt{\sigma + \phi^2[q(x)]} + \frac{E_i}{\phi^{i-1}[q(x)]\sqrt{\sigma + \phi^2[q(x)]}} \Big\} \qquad (3)$$

with

$$\phi' = \sigma + \phi^2, \tag{4}$$

where $A(x)$, $B_i(x)$, $C_i(x)$, $D_i(x)$, $E_i(x)$, $q(x)$ are functions of the indicated argument to be determined, σ is an arbitrary constant, and the prime denotes ϕ differentiation with respect to q. To determine u explicitly, one may substitute (3) and (4) into the given NPDE and collect coefficients of polynomials of ϕ, then eliminate each coefficient to derive a set of partial differential equations for A, B_i, C_i, $D_i(x)$, $E_i(x)$, and q, and solve the system of partial differential equations to obtain A, B_i, C_i, $D_i(x)$, $E_i(x)$, and q. Finally, as (4) possesses the general solutions

$$\phi = \begin{cases} -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}q), & \sigma < 0, \\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}q), & \sigma < 0, \\ \sqrt{\sigma}\tan(\sqrt{\sigma}q), & \sigma > 0, \\ -\sqrt{\sigma}\cot(\sqrt{\sigma}q), & \sigma > 0, \\ \dfrac{-1}{q}, & \sigma = 0, \end{cases} \tag{5}$$

and substituting A, B_i, C_i, $D_i(x)$, $E_i(x)$, q and (5) into (3), one can obtain the exact solutions to the given NPDE.

Now we apply the improved mapping approach to (1). By the balancing procedure, ansatz (3) becomes

$$u = f + g\phi(q) + h\sqrt{\sigma + \phi^2(q)} + \frac{A}{\phi(q)} + \frac{B}{\sqrt{\sigma + \phi^2(q)}}, \tag{6}$$

where f, g, h, A, B, and q are functions of (x, y, t) to be determined. Substituting (6) and (4) into (1) and collecting coefficients of polynomials of ϕ, then setting each coefficient to zero, we have

$$\begin{aligned} f &= \frac{1}{4}\int \mathrm{d}x\big[q_x^4 q_y \sigma - 2q_x^2 q_{xxy} + 2q_x q_{xy} q_{xx} \\ &\quad + q_{xx}^2 q_y - 2q_x q_y q_{xxx} + q_x^2 q_t\big](q_x^2 q_y)^{-1}, \\ g &= -q_x,\ h = q_x,\ A = -\frac{1}{2}\frac{q_x q_{xy}\sigma + q_y q_{xyt}}{q_x q_y}, \\ B &= \frac{1}{2}\frac{q_x q_{xy}\sigma + q_y q_{xyt}}{q_x q_y} \end{aligned} \tag{7}$$

with the function q in the variable separated form

$$q = \chi(x) + \varphi(y - ct), \tag{8}$$

where $\chi(x)$ and $\varphi(y - ct)$ are two arbitrary functions of the indicated arguments. Here c is an arbitrary constant. Based on the solutions of (4), one thus obtains an explicit solution of (1).

Case 1. For $\sigma = -1$, we can derive the following solitary wave solutions of (1):

$$\begin{aligned} u_1 &= -\frac{1}{4}\int \frac{\chi_x^4 - \chi_{xx}^2 + 2\chi_x\chi_{xxx} + c\chi_x^2}{\chi_x^2}\mathrm{d}x \\ &\quad + \chi_x\Big[\tanh(\chi + \varphi) + \sqrt{\tanh(\chi + \varphi)^2 - 1}\Big], \end{aligned} \tag{9}$$

$$\begin{aligned} u_2 &= -\frac{1}{4}\int \frac{\chi_x^4 - \chi_{xx}^2 + 2\chi_x\chi_{xxx} + c\chi_x^2}{\chi_x^2}\mathrm{d}x \\ &\quad + \chi_x\Big[\coth(\chi + \varphi) + \sqrt{\coth(\chi + \varphi)^2 - 1}\Big], \end{aligned} \tag{10}$$

with two arbitrary functions being $\chi(x)$ and $\varphi(y - ct)$.

Case 2. For $\sigma = 1$, we can obtain the following periodic wave solutions of (1):

$$\begin{aligned} u_3 &= \frac{1}{4}\int \frac{\chi_x^4 + \chi_{xx}^2 - 2\chi_x\chi_{xxx} - c\chi_x^2}{\chi_x^2}\mathrm{d}x \\ &\quad + \chi_x\Big[-\tan(\chi + \varphi) + \sqrt{\tan(\chi + \varphi)^2 + 1}\Big], \end{aligned} \tag{11}$$

$$\begin{aligned} u_4 &= \frac{1}{4}\int \frac{\chi_x^4 + \chi_{xx}^2 - 2\chi_x\chi_{xxx} - c\chi_x^2}{\chi_x^2}\mathrm{d}x \\ &\quad + \chi_x\Big[\cot(\chi + \varphi) + \sqrt{\cot(\chi + \varphi)^2 + 1}\Big], \end{aligned} \tag{12}$$

with two arbitrary functions being $\chi(x)$ and $\varphi(y - ct)$.

Case 3. For $\sigma = 0$, we can derive the following variable separated solution of (1):

$$u_5 = \frac{1}{4}\int \frac{\chi_{xx}^2 - 2\chi_x\chi_{xxx} - c\chi_x^2}{\chi_x^2}\mathrm{d}x + 2\frac{\chi_x}{\chi + \varphi}, \tag{13}$$

with two arbitrary functions being $\chi(x)$ and $\varphi(y - ct)$.

3. Some Novel Folded Localized Excitations

Now we will discuss some new types of folded localized excitations from the potential of the solitary wave solution determined by (10) in Case 1 and rewrite it in a simple form, namely

$$\begin{aligned} U = u_{2y} &= -\chi_x\varphi_y \mathrm{csch}(\chi + \varphi) \\ &\quad \cdot [\mathrm{csch}(\chi + \varphi) + \coth(\chi + \varphi)]. \end{aligned} \tag{14}$$

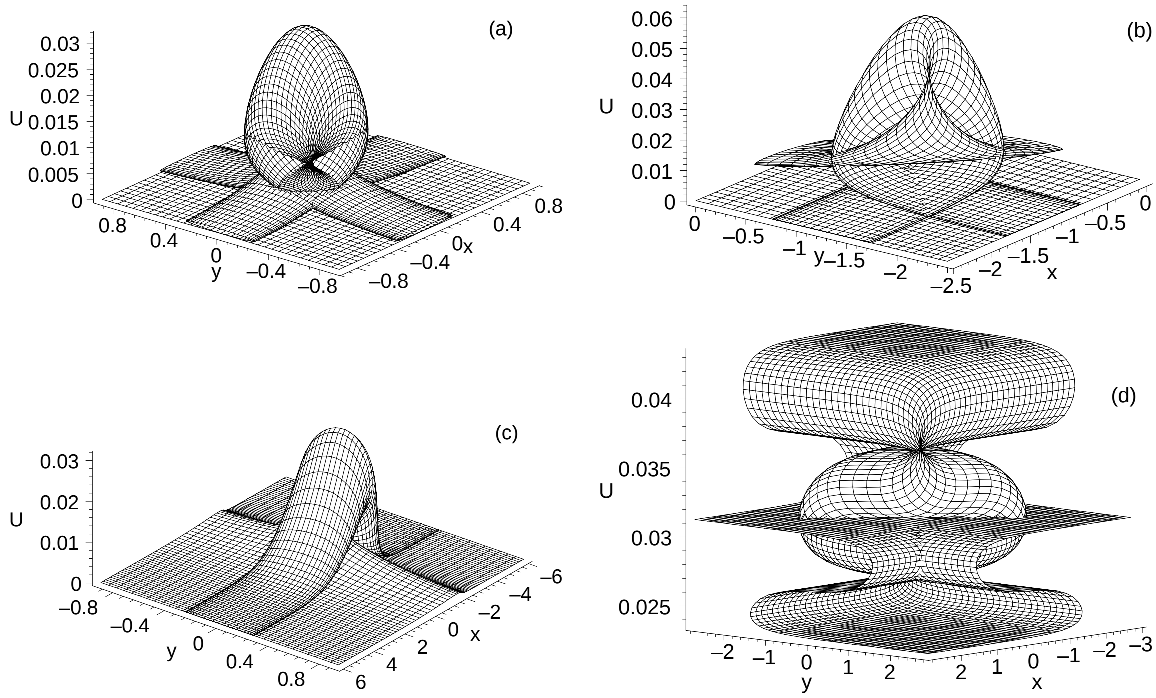


Fig. 1. Four types of folded localized excitation depicted for the field U determined by (14) at the time $t = 0$ with the choices: (a) $\chi_x = -\mathrm{sech}^2(\zeta)$, $x = \zeta + \tanh(\zeta)$, $\varphi_y = \mathrm{sech}^2(\varepsilon - ct)$, $y = \varepsilon + \tanh(\varepsilon)$; (b) χ_x and φ_y are same as in (a), however, $x = \zeta + 1.2\tanh(\zeta) + 1.2\mathrm{sech}(\zeta)$, $y = \varepsilon + 1.2\tanh(\varepsilon) + 1.2\mathrm{sech}(\varepsilon)$; (c) χ_x and φ_y are same as in (a), however, $x = \zeta - 1.8\mathrm{sech}(\zeta)$, $y = \varepsilon - 1.8\tanh(\varepsilon)$; (d) $\chi_x = -1.5\mathrm{sech}^2(\zeta) - 1.5\tanh^2(\zeta)$, $x = \zeta + 2\tanh(\zeta)$, $\varphi_y = 1.5\mathrm{sech}^2(\varepsilon - ct) + 1.5\tanh^2(\varepsilon - ct)$, $y = \varepsilon + 2\tanh(\varepsilon)$.

In order to construct kinds of interesting folded localized excitations, we introduce some suitable multi-valued functions [16]. For instance,

$$\begin{aligned} \varphi_y &= \sum_{j=1}^{M} V_j(\varepsilon - c_j t), \\ y &= \varepsilon + \sum_{j=1}^{M} P_j(\varepsilon - c_j t), \\ \varphi &= \int^{\varepsilon} \varphi_y y_\varepsilon \mathrm{d}\varepsilon, \end{aligned} \tag{15}$$

where V_j and P_j are localized excitations with the properties $V_j(\pm\infty) = 0$, $P_j(\pm\infty)$ = const. From (15), one knows that ε may be a multi-valued function in some suitable regions of y by selecting the function P_j appropriately. Therefore, the function φ_y, which is obviously an interaction solution of M localized excitations since the property $\varepsilon_{y\longrightarrow\infty} \longrightarrow \infty$, may be a multi-valued function of y in these areas, though it is a single-valued functions of ε. Similarly, we also treat the function $\chi(x)$ in this way:

$$\begin{aligned} \chi_x &= \sum_{j=1}^{N} U_j(\zeta), \quad x = \zeta + \sum_{j=1}^{N} Q_j(\zeta), \\ \chi &= \int^{\zeta} \chi_x x_\zeta \mathrm{d}\zeta. \end{aligned} \tag{16}$$

In Fig. 1, four types of folded localized excitation are presented for the field U determined by (14). The related functions chosen in the plots are directly given in the figure caption.

4. Some Localized Excitations with Chaotic Behaviours

Just as solitons, chaos is another important part of nonlinear science. It has been widely applied in many natural sciences. In this section, we mainly discuss some localized coherent excitations with chaotic behaviour in the (2+1)-dimensional breaking soliton system.

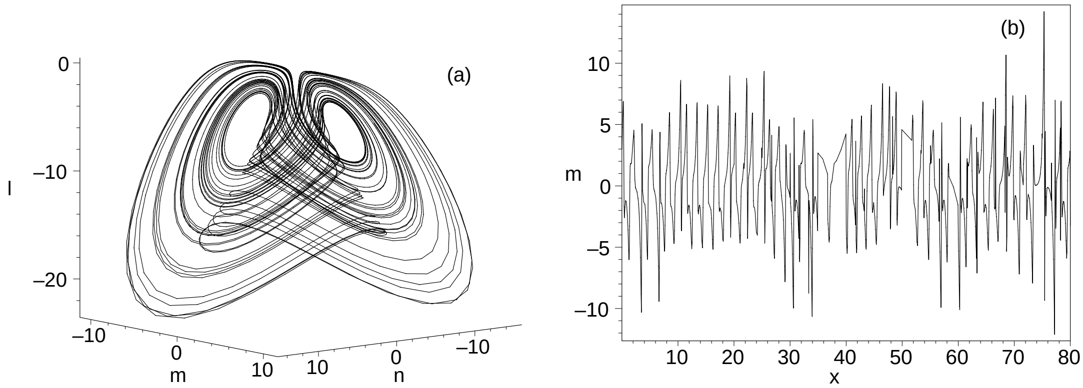


Fig. 2. (a) A novel butterfly-like attractor plot of the chaotic LCC system (17) with the initial condition (18). (b) A typical plot of the chaotic solution m of (17) related to (a).

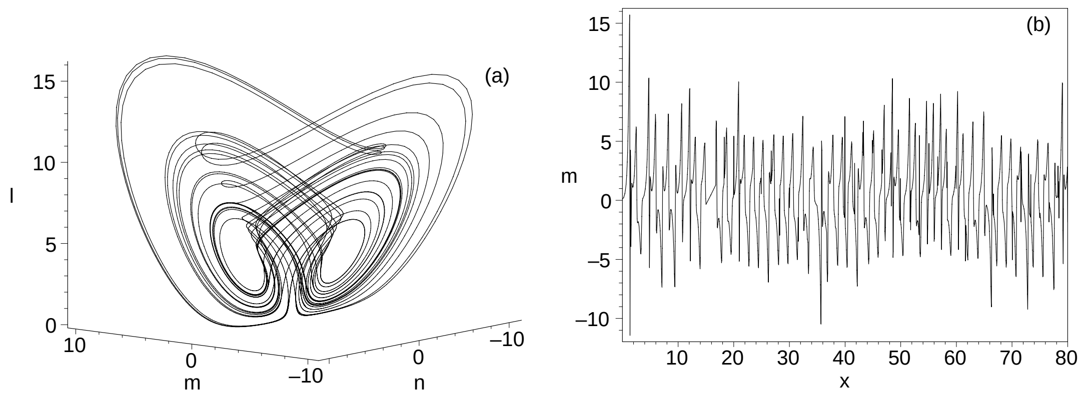


Fig. 3. (a) Another novel butterfly-like attractor plot of the chaotic LCC system (17) with the initial condition (19). (b) A typical plot of the chaotic solution m of (17) related to (a).

4.1. Novel Butterfly-Like Chaotic Attractor

Recently, Lü et al. [22] have introduced a new chaotic system (LCC system) of three-dimensional quadratic autonomous ordinary differential equations, which can display two 1-scroll chaotic attractors simultaneously with only three equilibria and two 2-scroll chaotic attractors simultaneously with five equilibria [23]:

$$m_\xi = -12m + ln, \quad n_\xi = -5n + ml, \quad l_\xi = 4.5l - mn, \tag{17}$$

where m, n, and l are functions of ξ ($\xi = x$ or $\xi = y - ct$). A novel butterfly-like chaotic attractor for the LCC system (17) is depicted in Fig. 2, when

$$m(0) = 2, \quad n(0) = -3, \quad l(0) = 2. \tag{18}$$

The shape of the chaotic attractor will be changed, when the initial conditions are altered. For example, if we change the initial conditions of the LCC system as follows:

$$m(0) = 0.1, \quad n(0) = 0.1, \quad l(0) = 0.1, \tag{19}$$

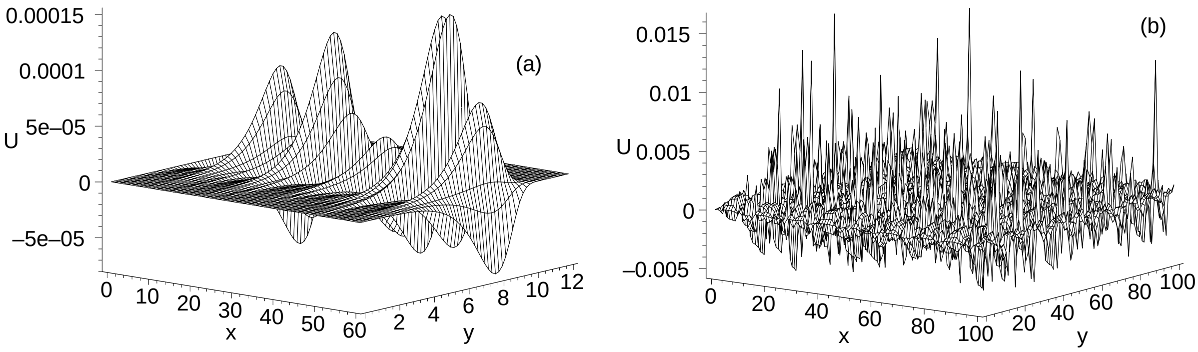
we can obtain another novel butterfly-like chaotic attractor shown in Figure 3.

4.2. Chaotic Patterns

If the functions χ and/or φ are assumed to be solutions of a chaotic dynamical system, we can derive some localized excitations with chaotic behaviour. For example, χ is defined to be a solution of the LCC system (17), and take

$$\chi = 1+0.01m(x), \quad \varphi = 1+0.01\exp(y-ct), \tag{20}$$

where $m(x)$ is a solution of the LCC system (17) with the initial conditions (19). By this choice, the dromion



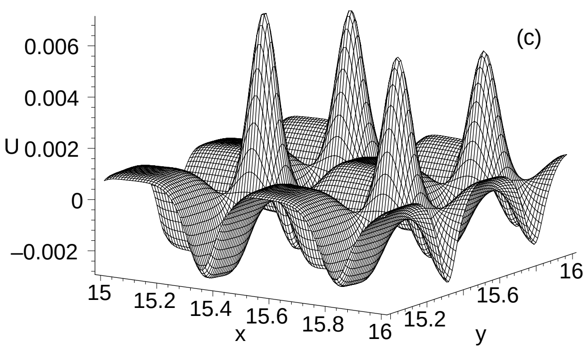


Fig. 4. (a) Plot of the chaotic line soliton for the field U determined by (14) with the condition (20) and $c = 1$, $t = 0$. (b) Plot of the chaotic pattern for the field U expressed by (14) with the condition (21) and $c = 1$, $t = 0$. (c) An enlargement of the centre area ($x \in [15, 16]$, $y \in [15, 16]$) of (b).

localized in all directions is changed into a chaotic line soliton, which presents chaotic behaviour in the x-direction, though still localized in the y-direction. Figure 4a shows the corresponding plot of the chaotic line soliton for the field U of (14) with parameter $c = 1$ at time $t = 0$.

Furthermore, if χ and φ are all selected as chaotic solutions of the LCC system, the field U of (14) will behave chaotically in all directions and will yield a chaotic pattern. For example, χ and φ may be chosen as

$$\chi(x) = 1 + m(x), \quad \varphi(y) = 1 + m(y - ct), \tag{21}$$

where $m(x)$ and $m(y-ct)$ are the solutions of the LCC system (17) with the initial conditions (19). Figure 4b shows a plot of the special chaotic pattern for the field U expressed by (14) with the condition (21) at time $t = 0$. In order to show that the chaotic behaviour is due to the peak value of solitons, we enlarge a small region ($x \in [15, 16]$, $y \in [15, 16]$) of Figure 4b. The result is shown in Fig. 4c, which clearly presents a kind of dromion with a chaotic structure.

5. Summary and Discussion

In summary, via an improved mapping approach and a linear variable separation approach, the (2+1)-dimensional breaking soliton system is solved. Abundant localized coherent soliton structures of the solution U of (14) like dromions, peakons, breathers, instantons, can be easily constructed by selecting appropriate arbitrary functions. Except for the single-valued localized excitation, we find a new type of multi-valued localized excitation, i. e. folded solitary wave and/or foldon excitation for the (2+1)-dimensional breaking soliton system. To our knowledge, the folded solitary wave and/or foldon excitation for the (2+1)-dimensional breaking soliton system have not been reported in the previous literature.

Additionally, using the nuclear spin generator (NSG) chaotic system, Fang and Zheng [24] have recently obtained some chaotic solitons of the (2+1)-dimensional generalized Broer-Kaup system. Along the above line, we use the LCC chaotic system to get some new chaotic solutions, which are different from the ones presented in the previous work. Since the wide applications of the soliton theory, to learn more about the localized excitations and their applications in reality is worth to be studied further.

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